An approach to the exterior Dirichlet problem in R^2 with applications to wave scattering[1]

The study of phenomena arising from the scattering of waves from rough surfaces is of interest in many fields of physics and engineering. For instance, one of the most promising application in optics seems to be the employment of a laser beam scattering for non-destructive inspection of topography and optical quality of surfaces.

In dealing with scattering from rough surfaces, approximate methods, such as perturbation techniques [1] or the Kirchhoff approximation [2], have gained ground. Unfortunately, the validity of these methods proves insufficient to handle interaction with resonance roughnesses, as well as grazing incidence geometries. For this reason a large amount of research, reported in the last 15 years, has been devoted to rigorous methods based, in particular, on boundary integral equations. (A review may be found in [3].)

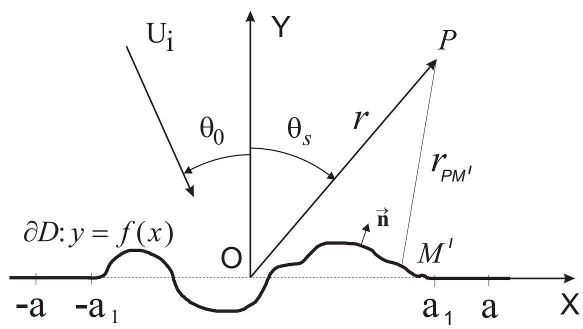

Fig. 1. Geometry of the problem.

In this paper we investigate the external 2-D Dirichlet boundary value problem (BVP) for the Helmholtz equation:

$$\Delta U(P) + k^2 U(P) = 0, \quad U|_{\partial D} = 0, \tag{1}$$

where U is the total field, $P = (x, y) \in \mathbf{R}^2$, $k = \frac{2\pi}{\lambda} \in \mathbf{R}$ is the wave number, λ is the wavelength, $\partial D = \{(x, f(x)) : x \in \mathbf{R}\}$ with $f \in \mathbf{C}^2$ and $\operatorname{supp} f \subseteq [-a_1, a_1]$ (see Fig. 1). This BVP determines the reflection of an electromagnetic harmonic TE-polarized wave from a 1-D perfectly conducting rough surface with boundary ∂D, as well as the scattering by a sound-soft wall in acoustics.

It is convenient to represent (1) in the equivalent form

$$\Delta U_s(P) + k^2 U_s(P) = 0, \quad U_s|_{\partial D} = -(U_i + U_m)|_{\partial D}, \tag{2}$$

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where $U_s = U - U_i - U_m$ is the scattered field, $U_i = \exp[i(\alpha_0 x - \beta_0 y)]$ is the incident plane wave, $U_m = -\exp[i(\alpha_0 x + \beta_0 y)]$ is the mirror-reflected wave with respect to the plane $y = 0$, $\alpha_0 = k\sin\theta_0$, $\beta_0 = k\cos\theta_0$ and θ_0 is the angle of incidence. If the radiation condition on U_s is imposed, then BVP (2) has a unique solution [4].

An attempt to solve BVP (2) was made in [5], where an integral equation of the first kind on an unbounded domain with a weakly singular kernel for an unknown distribution of surface current density was obtained. In order to solve that equation, it was suggested that the integration domain should be truncated, which seems untenable from the mathematical and physical points of view. We introduce a new rigorous formalism, which reduces the problem to Fredholm integral equations of the second kind and ensures a high accuracy and stability of computations.

We seek the solution of (2) as a double layer potential

$$U_s(P) = \int_{\partial D} \mu(M') \frac{\partial H_0^{(1)}(kr_{PM'})}{\partial n'}\, ds', \tag{3}$$

where $\dfrac{\partial H_0^{(1)}(kr_{PM'})}{\partial n'}$ is the normal derivative of the Hankel function of the first kind and order zero, $r_{PM'}$ is the distance between the observation point P and $M' \in \partial D$, ds' is an arc length along ∂D and $\mu(M')$ is the density of the scattered field sources to be found.

The integral equation for $\mu(x)$ can be obtained from (3) by passing to the limit as $P \to M \in \partial D$:

$$\mu(x) + \int_{-\infty}^{\infty} K(x, x')\, \mu(x')\, dx' = g(x), \tag{4}$$

where the kernel $K(x, x') = \dfrac{ik}{2} \dfrac{H_1^{(1)}(kr_{MM'})}{r_{MM'}} [f(x) - f(x') - f'(x')(x - x')]$ is continuous, and $g(x) = -\exp(i\alpha_0 x)\sin[\beta_0 f(x)]$. We reduce (4) to a Fredholm integral equation of the second kind. For this purpose we split $\mu(x)$, writing $\mu(x) = \mu_I(x) + \mu_E(x)$, where

$$\mu_I(x) \equiv \begin{cases} \mu(x), & |x| < a \\ 0, & |x| > a \end{cases}, \qquad \mu_E(x) \equiv \begin{cases} 0, & |x| < a \\ \mu(x), & |x| > a \end{cases}, \qquad a > a_1. \tag{5}$$

Further, noting that $K(x, x') = 0$ for any $|x|, |x'| > a_1$, we can transform (4) into an equation for the internal density μ_I:

$$\mu_I(x) + \int_{|x'|<a} [K(x, x') + \tilde{K}(x, x')]\, \mu_I(x')\, dx' = g(x), \qquad |x| < a, \tag{6}$$

where

$$\tilde{K}(x, x') = -\int_{|x''|>a} K(x, x'')K(x'', x')\, dx'' \tag{7}$$

and the external density μ_E is expressed in terms of the internal one:

$$\mu_E(x) = -\int_{|x'|<a_1} K(x, x')\mu_I(x')\, dx'. \tag{8}$$

It is worth noting that setting the width $a-a_1$ of the transient zone large enough, we can use integration by parts to obtain an asymptotic estimate for $\tilde{K}(x,x')$. Thus, solving (6), for example, by a moment method and using (8), the density $\mu(x)$ can be calculated for any x.

Once the density μ has been obtained, the scattered field U_s can be deduced from (3). We now write (3) in the polar co-ordinate system (r,θ_s) as

$$U_s(r,\theta_s) = [2\pi/(kr)]^{1/2}\ \exp[i(kr-\pi/4)]A(\theta_s,\theta_0) + O[(kr)^{-3/2}],$$

where

$$A(\theta_s,\theta_0) = \frac{i}{\pi}\int_{-\infty}^{\infty}[\alpha_s f'(x) - \beta_s]\exp\{-i[\alpha_s x + \beta_s f(x)]\}\mu(x)\,dx \tag{9}$$

is the scattering amplitude, $\alpha_s = k\sin\theta_s$ and $\beta_s = k\cos\theta_s$. Substituting (5) and (7) into (9), we represent $A(\theta_s,\theta_0)$ as $A(\theta_s,\theta_0) = A_I(\theta_s,\theta_0)+A_E(\theta_s,\theta_0)$, where $A_I(\theta_s,\theta_0)$ is given by (9) with the truncated integration domain $[-a,a]$. As for A_E, after cumbersome calculations we find that, in terms of μ_I,

$$\begin{aligned}
A_E(\theta_s,\theta_0) = &-\frac{k}{\pi(2\pi i)^{1/2}}\int_{|x|<a_1}\mu_I(x)\exp(-i\alpha_s x)\\
&\times\Bigg\{\beta_s\left(if(x)+\frac{\varepsilon f'(x)}{8k}\right)\left[\frac{SI_1(\Lambda^{(+)},b^{(+)},\varepsilon)}{(\Lambda^{(+)})^{1/2}}+\frac{SI_1(\Lambda^{(-)},b^{(-)},-\varepsilon)}{(\Lambda^{(-)})^{1/2}}\right]+\\
&\frac{\beta_s f'(x)}{k}\Bigg[\frac{1-i[8\Lambda^{(+)}(1+[b^{(+)}]^2)^{1/2}]^{-1}}{[\Lambda^{(+)}(1+[b^{(+)}]^2)^{1/2}]^{1/2}}\exp\{i\Lambda^{(+)}[(1+[b^{(+)}]^2)^{1/2}+\varepsilon]\}-\\
&\frac{1-i[8\Lambda^{(-)}(1+[b^{(-)}]^2)^{1/2}]^{-1}}{[\Lambda^{(-)}(1+[b^{(-)}]^2)^{1/2}]^{1/2}}\exp\{i\Lambda^{(-)}[(1+[b^{(-)}]^2)^{1/2}-\varepsilon]\}\Bigg]\\
&+i\varepsilon f'(x)\{[(1+\varepsilon)\Lambda^{(+)}]^{1/2}SI_0(\Lambda^{(+)},b^{(+)},\varepsilon)+[(1-\varepsilon)\Lambda^{(-)}]^{1/2}SI_0(\Lambda^{(-)},b^{(-)},-\varepsilon)\}\Bigg\}\,dx\\
&+o([k(a-a_1)]^{-3/2}),
\end{aligned}\tag{10}$$

where $\varepsilon = \sin\theta_s$, $\Lambda^{(\pm)} = k(a\pm x)$, $b^{(\pm)} = f(x)/(a\pm x)$, and

$$\begin{aligned}
SI_0(\Lambda,b,\varepsilon) &= (1+\varepsilon)^{1/2}\int_1^{\infty}\frac{\exp\{i\Lambda[(t^2+b^2)^{1/2}+\varepsilon t]\}}{(t^2+b^2)^{1/4}}\,dt,\\
SI_1(\Lambda,b,\varepsilon) &= \int_1^{\infty}\frac{\exp\{i\Lambda[(t^2+b^2)^{1/2}+\varepsilon t]\}}{(t^2+b^2)^{3/4}}\,dt,\quad \Lambda\gg 0,\quad b\in\mathbf{R},\quad |\varepsilon|\le 1.
\end{aligned}\tag{11}$$

The integrals (11) can be evaluated by means of the uniform asymptotic technique [6].

The calculations of $A(\theta_s,\theta_0)$ have been tested against the energy balance criterion

(EBC) [5]:

$$\Delta\mathcal{E} = \left| 1 - [2\mathrm{Re}A(\theta_0, \theta_0)]^{-1} \cdot \int_{-\pi/2}^{\pi/2} |A(\theta_s, \theta_0)|^2 \, d\theta_s \right| \equiv 0,$$

which is closely connected with the relative error of computation. We note that $A(\pm\pi/2, \theta_0) \equiv 0$.

To illustrate the results obtained, we consider the scattering from a finite deep resonance grating

$$f(x) = h\eta(x)\cos(2\pi x/d), \tag{12}$$

where $\eta(x)$ is the cutting function [7], $h = d = \lambda$, $a = 25\lambda$ and $a_1 = 20\lambda$ (see Fig. 2). The number of matching points for solving (6) was 1000. In Fig. 3 the energy defect $\Delta\mathcal{E}$ (in percentage) versus the angle of incidence for the integral equation method discussed (solid line) and offered in [5] (broken line) is plotted. It is obvious that the scattering amplitude calculated following [5] is incompatible with EBC and, thus, should be rejected. On the other hand, the proposed method yields very accurate data ($\max \Delta\mathcal{E} < 0.3\%$). Similar conclusions are valid for large-scale, steep, smooth or resonance modulated asperities, including the grazing incidence geometry. Figs. 4–6 show the modulus of the density $\mu(x)$ and the corresponding scattered field intensity at infinity $|A(\theta_s, \theta_0)|^2$ for angles of incidence $\theta_0 = 0^\circ$ (Fig. 4), $\theta_0 = 30^\circ$ (Fig. 5) and $\theta_0 = 89^\circ$ (Fig. 6). We note that for a sufficiently shallow roughness (for example, for (12) with $h < 0.2\lambda$) and quasinormal incidence both approaches give qualitatively resembling results, though the error of our calculations is, as a rule, smaller by two orders of magnitude.

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