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1 Introduction

This paper contains examples of various features from $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

2 Enumeration of Hamiltonian paths in a graph

Let $\mathbf{A} = (a_{ij})$ be the adjacency matrix of graph G. The corresponding Kirchhoff matrix $\mathbf{K} = (k_{ij})$ is obtained from $\mathbf{A}$ by replacing in $-\mathbf{A}$ each diagonal entry by the degree of its corresponding vertex; i.e., the ith diagonal entry is identified with the degree of the ith vertex. It is well known that

$$\det\mathbf{K}(i|i) = \text{ the number of spanning trees of } G, \quad i = 1,\dots,n \tag{1}$$

where $\mathbf{K}(i|i)$ is the ith principal submatrix of $\mathbf{K}$.

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\det\mathbf{K}(i|i)=\text{ the number of spanning trees of $G$},
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Let $C_{i(j)}$ be the set of graphs obtained from G by attaching edge $(v_i v_j)$ to each spanning tree of G. Denote by $C_i = \bigcup_j C_{i(j)}$. It is obvious that the collection of Hamiltonian cycles is a subset of C_i. Note that the cardinality of C_i is $k_{ii}\det\mathbf{K}(i|i)$. Let $\widehat{X} = \{\hat{x}_1,\dots,\hat{x}_n\}$.

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$\wh X=\{\hat x_1,\dots,\hat x_n\}$
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Define multiplication for the elements of $\widehat{X}$ by

$$\hat{x}_i\hat{x}_j = \hat{x}_j\hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i,j = 1,\dots,n. \tag{2}$$

Let $\hat{k}_{ij} = k_{ij}\hat{x}_j$ and $\hat{k}_{ij} = -\sum_{j\neq i}\hat{k}_{ij}$. Then the number of Hamiltonian cycles H_c is given by the relation [8]

$$\biggl(\prod_{j=1}^{n}\hat{x}_j\biggr)H_c = \frac{1}{2}\hat{k}_{ij}\det\widehat{\mathbf{K}}(i|i), \qquad i = 1,\dots,n. \tag{3}$$

The task here is to express (3) in a form free of any $\hat{x}_i$, $i = 1, \dots, n$. The result also leads to the resolution of enumeration of Hamiltonian paths in a graph.

It is well known that the enumeration of Hamiltonian cycles and paths in a complete graph K_n and in a complete bipartite graph $K_{n_1 n_2}$ can only be found from *first combinatorial principles* [4]. One wonders if there exists a formula which can be used very efficiently to produce K_n and $K_{n_1 n_2}$. Recently, using Lagrangian methods, Goulden and Jackson have shown that H_c can be expressed in terms of the determinant and permanent of the adjacency matrix [3]. However, the formula of Goulden and Jackson determines neither K_n nor $K_{n_1 n_2}$ effectively. In this paper, using an algebraic method, we parametrize the adjacency matrix. The resulting formula also involves the determinant and permanent, but it can easily be applied to K_n and $K_{n_1 n_2}$. In addition, we eliminate the permanent from H_c and show that H_c can be represented by a determinantal function of multivariables, each variable with domain $\{0, 1\}$. Furthermore, we show that H_c can be written by number of spanning trees of subgraphs. Finally, we apply the formulas to a complete multigraph $K_{n_1 \dots n_p}$.

The conditions $a_{ij} = a_{ji}$, $i, j = 1, \dots, n$, are not required in this paper. All formulas can be extended to a digraph simply by multiplying H_c by 2.

3 Main Theorem

Notation. For $p, q \in P$ and $n \in \omega$ we write $(q, n) \le (p, n)$ if $q \le p$ and $A_{q,n} = A_{p,n}$.

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\begin{notation} For $p,q\in P$ and $n\in\omega$
...
\end{notation}
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Let $\mathbf{B} = (b_{ij})$ be an $n \times n$ matrix. Let $\mathbf{n} = \{1, \dots, n\}$. Using the properties of (2), it is readily seen that

Lemma 3.1.

$$\prod_{i\in\mathbf{n}} \Bigl(\sum_{j\in\mathbf{n}} b_{ij}\hat{x}_i\Bigr) = \Bigl(\prod_{i\in\mathbf{n}} \hat{x}_i\Bigr) \operatorname{per} \mathbf{B} \tag{4}$$

where $\operatorname{per}\mathbf{B}$ *is the permanent of* $\mathbf{B}$.

Let $\widehat{Y} = \{\hat{y}_1, \dots, \hat{y}_n\}$. Define multiplication for the elements of $\widehat{Y}$ by

$$\hat{y}_i\hat{y}_j + \hat{y}_j\hat{y}_i = 0, \quad i, j = 1, \dots, n. \tag{5}$$

Then, it follows that

Lemma 3.2.

$$\prod_{i\in\mathbf{n}} \Bigl(\sum_{j\in\mathbf{n}} b_{ij}\hat{y}_j\Bigr) = \Bigl(\prod_{i\in\mathbf{n}} \hat{y}_i\Bigr) \det \mathbf{B}. \tag{6}$$

Note that all basic properties of determinants are direct consequences of Lemma 3.2. Write

$$\sum_{j\in\mathbf{n}} b_{ij}\hat{y}_j = \sum_{j\in\mathbf{n}} b_{ij}^{(\lambda)}\hat{y}_j + (b_{ii} - \lambda_i)\hat{y}_i\hat{y} \tag{7}$$

where

$$b_{ii}^{(\lambda)} = \lambda_i, \quad b_{ij}^{(\lambda)} = b_{ij}, \quad i \neq j. \tag{8}$$

Let $\mathbf{B}^{(\lambda)} = (b_{ij}^{(\lambda)})$. By (6) and (7), it is straightforward to show the following result:

Theorem 3.3.

$$\det \mathbf{B} = \sum_{l=0}^{n} \sum_{I_l \subseteq n} \prod_{i \in I_l} (b_{ii} - \lambda_i) \det \mathbf{B}^{(\lambda)}(I_l | I_l), \tag{9}$$

where $I_l = \{i_1, \dots, i_l\}$ and $\mathbf{B}^{(\lambda)}(I_l|I_l)$ is the principal submatrix obtained from $\mathbf{B}^{(\lambda)}$ by deleting its $i_1, \dots, i_l$ rows and columns.

Remark 3.1. Let $\mathbf{M}$ be an $n \times n$ matrix. The convention $\mathbf{M}(\mathbf{n}|\mathbf{n}) = 1$ has been used in (9) and hereafter.

Before proceeding with our discussion, we pause to note that Theorem 3.3 yields immediately a fundamental formula which can be used to compute the coefficients of a characteristic polynomial [9]:

Corollary 3.4. *Write* $\det(\mathbf{B} - x\mathbf{I}) = \sum_{l=0}^{n} (-1)^l b_l x^l$. *Then*

$$b_l = \sum_{I_l \subseteq \mathbf{n}} \det \mathbf{B}(I_l | I_l). \tag{10}$$

Let

$$\mathbf{K}(t, t_1, \dots, t_n) = \begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \dots & \dots & \dots & \dots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix}, \tag{11}$$

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\begin{pmatrix} D_1t&-a_{12}t_2&\dots&-a_{1n}t_n\\
-a_{21}t_1&D_2t&\dots&-a_{2n}t_n\\
\hdotsfor[2]{4}\\
-a_{n1}t_1&-a_{n2}t_2&\dots&D_nt\end{pmatrix}
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where

$$D_i = \sum_{j \in \mathbf{n}} a_{ij} t_j, \quad i = 1, \dots, n. \tag{12}$$

Set

$$D(t_1, \dots, t_n) = \frac{\delta}{\delta t} \det \mathbf{K}(t, t_1, \dots, t_n)|_{t=1}.$$

Then

$$D(t_1, \dots, t_n) = \sum_{i \in \mathbf{n}} D_i \det \mathbf{K}(t = 1, t_1, \dots, t_n; i|i), \tag{13}$$

where $\mathbf{K}(t = 1, t_1, \dots, t_n; i|i)$ is the ith principal submatrix of $\mathbf{K}(t = 1, t_1, \dots, t_n)$.

Theorem 3.3 leads to

$$\det \mathbf{K}(t_1, t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|} t^{n-|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda t)}(\overline{I}|\overline{I}). \tag{14}$$