

Sample Paper for the amsmath Package

File name: testmath.tex

American Mathematical Society

Version 2.0, 1999/11/15

1 Introduction

This paper contains examples of various features from $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

2 Enumeration of Hamiltonian paths in a graph

Let $\mathbf{A} = (a_{ij})$ be the adjacency matrix of graph G . The corresponding Kirchhoff matrix $\mathbf{K} = (k_{ij})$ is obtained from \mathbf{A} by replacing in $-\mathbf{A}$ each diagonal entry by the degree of its corresponding vertex; i.e., the i th diagonal entry is identified with the degree of the i th vertex. It is well known that

$$\det \mathbf{K}(i|i) = \text{the number of spanning trees of } G, \quad i = 1, \dots, n \quad (1)$$

where $\mathbf{K}(i|i)$ is the i th principal submatrix of \mathbf{K} .

$\backslash\det\mathbf{K}(i|i)=\text{the number of spanning trees of } G$,

Let $C_{i(j)}$ be the set of graphs obtained from G by attaching edge $(v_i v_j)$ to each spanning tree of G . Denote by $C_i = \bigcup_j C_{i(j)}$. It is obvious that the collection of Hamiltonian cycles is a subset of C_i . Note that the cardinality of C_i is $k_{ii} \det \mathbf{K}(i|i)$. Let $\widehat{X} = \{\hat{x}_1, \dots, \hat{x}_n\}$.

$\backslash\wh X=\{\hat{x}_1, \dots, \hat{x}_n\}$

Define multiplication for the elements of \widehat{X} by

$$\hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i, j = 1, \dots, n. \quad (2)$$

Let $\hat{k}_{ij} = k_{ij} \hat{x}_j$ and $\hat{k}_{ij} = -\sum_{j \neq i} \hat{k}_{ij}$. Then the number of Hamiltonian cycles H_c is given by the relation [8]

$$\left(\prod_{j=1}^n \hat{x}_j \right) H_c = \frac{1}{2} \hat{k}_{ij} \det \widehat{\mathbf{K}}(i|i), \quad i = 1, \dots, n. \quad (3)$$

The task here is to express (3) in a form free of any \hat{x}_i , $i = 1, \dots, n$. The result also leads to the resolution of enumeration of Hamiltonian paths in a graph.

It is well known that the enumeration of Hamiltonian cycles and paths in a complete graph K_n and in a complete bipartite graph $K_{n_1 n_2}$ can only be found from *first combinatorial principles* [4]. One wonders if there exists a formula which can be used very efficiently to produce K_n and $K_{n_1 n_2}$. Recently, using Lagrangian methods, Goulden and Jackson have shown that H_c can be expressed in terms of the determinant and permanent of the adjacency matrix [3]. However, the formula of Goulden and Jackson determines neither K_n nor $K_{n_1 n_2}$ effectively. In this paper, using an algebraic method, we parametrize the adjacency matrix. The resulting formula also involves the determinant and permanent, but it can easily be applied to K_n and $K_{n_1 n_2}$. In addition, we eliminate the permanent from H_c and show that H_c can be represented by a determinantal function of multivariables, each variable with domain $\{0, 1\}$. Furthermore, we show that H_c can be written by number of spanning trees of subgraphs. Finally, we apply the formulas to a complete multigraph $K_{n_1 \dots n_p}$.

The conditions $a_{ij} = a_{ji}$, $i, j = 1, \dots, n$, are not required in this paper. All formulas can be extended to a digraph simply by multiplying H_c by 2.

3 Main Theorem

Notation. For $p, q \in P$ and $n \in \omega$ we write $(q, n) \leq (p, n)$ if $q \leq p$ and $A_{q,n} = A_{p,n}$.

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\begin{notation} For  $p, q \in P$  and  $n \in \omega$ 
...
\end{notation}
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Let $\mathbf{B} = (b_{ij})$ be an $n \times n$ matrix. Let $\mathbf{n} = \{1, \dots, n\}$. Using the properties of (2), it is readily seen that

Lemma 3.1.

$$\prod_{i \in \mathbf{n}} \left(\sum_{j \in \mathbf{n}} b_{ij} \hat{x}_i \right) = \left(\prod_{i \in \mathbf{n}} \hat{x}_i \right) \text{per } \mathbf{B} \quad (4)$$

where $\text{per } \mathbf{B}$ is the permanent of \mathbf{B} .

Let $\hat{Y} = \{\hat{y}_1, \dots, \hat{y}_n\}$. Define multiplication for the elements of \hat{Y} by

$$\hat{y}_i \hat{y}_j + \hat{y}_j \hat{y}_i = 0, \quad i, j = 1, \dots, n. \quad (5)$$

Then, it follows that

Lemma 3.2.

$$\prod_{i \in \mathbf{n}} \left(\sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j \right) = \left(\prod_{i \in \mathbf{n}} \hat{y}_i \right) \det \mathbf{B}. \quad (6)$$

Note that all basic properties of determinants are direct consequences of Lemma 3.2. Write

$$\sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j = \sum_{j \in \mathbf{n}} b_{ij}^{(\lambda)} \hat{y}_j + (b_{ii} - \lambda_i) \hat{y}_i \hat{y} \quad (7)$$

where

$$b_{ii}^{(\lambda)} = \lambda_i, \quad b_{ij}^{(\lambda)} = b_{ij}, \quad i \neq j. \quad (8)$$

Let $\mathbf{B}^{(\lambda)} = (b_{ij}^{(\lambda)})$. By (6) and (7), it is straightforward to show the following result:

Theorem 3.3.

$$\det \mathbf{B} = \sum_{l=0}^n \sum_{I_l \subseteq \mathbf{n}} \prod_{i \in I_l} (b_{ii} - \lambda_i) \det \mathbf{B}^{(\lambda)}(I_l | I_l), \quad (9)$$

where $I_l = \{i_1, \dots, i_l\}$ and $\mathbf{B}^{(\lambda)}(I_l | I_l)$ is the principal submatrix obtained from $\mathbf{B}^{(\lambda)}$ by deleting its i_1, \dots, i_l rows and columns.

Remark 3.1. Let \mathbf{M} be an $n \times n$ matrix. The convention $\mathbf{M}(\mathbf{n} | \mathbf{n}) = 1$ has been used in (9) and hereafter.

Before proceeding with our discussion, we pause to note that Theorem 3.3 yields immediately a fundamental formula which can be used to compute the coefficients of a characteristic polynomial [9]:

Corollary 3.4. Write $\det(\mathbf{B} - x\mathbf{I}) = \sum_{l=0}^n (-1)^l b_l x^l$. Then

$$b_l = \sum_{I_l \subseteq \mathbf{n}} \det \mathbf{B}(I_l | I_l). \quad (10)$$

Let

$$\mathbf{K}(t, t_1, \dots, t_n) = \begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \dots & \dots & \dots & \dots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix}, \quad (11)$$

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where

$$D_i = \sum_{j \in \mathbf{n}} a_{ij} t_j, \quad i = 1, \dots, n. \quad (12)$$

Set

$$D(t_1, \dots, t_n) = \frac{\delta}{\delta t} \det \mathbf{K}(t, t_1, \dots, t_n) \Big|_{t=1}.$$

Then

$$D(t_1, \dots, t_n) = \sum_{i \in \mathbf{n}} D_i \det \mathbf{K}(t = 1, t_1, \dots, t_n; i | i), \quad (13)$$