

# Sample Paper for the `amsmath` Package

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### 1 Introduction

This paper contains examples of various features from  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$ .

### 2 Enumeration of Hamiltonian paths in a graph

Let  $\mathbf{A} = (a_{ij})$  be the adjacency matrix of graph  $G$ . The corresponding Kirchhoff matrix  $\mathbf{K} = (k_{ij})$  is obtained from  $\mathbf{A}$  by replacing in  $-\mathbf{A}$  each diagonal entry by the degree of its corresponding vertex; i.e., the  $i$ th diagonal entry is identified with the degree of the  $i$ th vertex. It is well known that

$$\det \mathbf{K}(i|i) = \text{the number of spanning trees of } G, \quad i = 1, \dots, n \quad (1)$$

where  $\mathbf{K}(i|i)$  is the  $i$ th principal submatrix of  $\mathbf{K}$ .

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Let  $C_{i(j)}$  be the set of graphs obtained from  $G$  by attaching edge  $(v_i v_j)$  to each spanning tree of  $G$ . Denote by  $C_i = \bigcup_j C_{i(j)}$ . It is obvious that the collection of Hamiltonian cycles is a subset of  $C_i$ . Note that the cardinality of  $C_i$  is  $k_{ii} \det \mathbf{K}(i|i)$ . Let  $\hat{X} = \{\hat{x}_1, \dots, \hat{x}_n\}$ .

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Define multiplication for the elements of  $\hat{X}$  by

$$\hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i, j = 1, \dots, n. \quad (2)$$

Let  $\hat{k}_{ij} = k_{ij} \hat{x}_j$  and  $\hat{k}_{ij} = -\sum_{j \neq i} \hat{k}_{ij}$ . Then the number of Hamiltonian cycles  $H_c$  is given by the relation [8]

$$\left( \prod_{j=1}^n \hat{x}_j \right) H_c = \frac{1}{2} \hat{k}_{ij} \det \hat{\mathbf{K}}(i|i), \quad i = 1, \dots, n. \quad (3)$$

The task here is to express (3) in a form free of any  $\hat{x}_i$ ,  $i = 1, \dots, n$ . The result also leads to the resolution of enumeration of Hamiltonian paths in a graph.

It is well known that the enumeration of Hamiltonian cycles and paths in a complete graph  $K_n$  and in a complete bipartite graph  $K_{n_1, n_2}$  can only be found from *first combinatorial principles* [4]. One wonders if there exists a formula which can be used very efficiently to produce  $K_n$  and  $K_{n_1, n_2}$ . Recently, using Lagrangian methods, Goulden and Jackson have shown that  $H_c$  can be expressed in terms of the determinant and permanent of the adjacency matrix [3]. However, the formula of Goulden and Jackson determines neither  $K_n$  nor  $K_{n_1, n_2}$  effectively. In this paper, using an algebraic method, we parametrize the adjacency matrix. The resulting formula also involves the determinant and permanent, but it can easily be applied to  $K_n$  and  $K_{n_1, n_2}$ . In addition, we eliminate the permanent from  $H_c$  and show that  $H_c$  can be represented by a determinantal function of multivariables, each variable with domain  $\{0, 1\}$ . Furthermore, we show that  $H_c$  can be written by number of spanning trees of subgraphs. Finally, we apply the formulas to a complete multigraph  $K_{n_1 \dots n_p}$ .

The conditions  $a_{ij} = a_{ji}$ ,  $i, j = 1, \dots, n$ , are not required in this paper. All formulas can be extended to a digraph simply by multiplying  $H_c$  by 2.

### 3 Main Theorem

*Notation.* For  $p, q \in P$  and  $n \in \omega$  we write  $(q, n) \leq (p, n)$  if  $q \leq p$  and  $A_{q, n} = A_{p, n}$ .

`\begin{notation} For  $p, q \in P$  and  $n \in \omega$`   
`...`  
`\end{notation}`

Let  $\mathbf{B} = (b_{ij})$  be an  $n \times n$  matrix. Let  $\mathbf{n} = \{1, \dots, n\}$ . Using the properties of (2), it is readily seen that

**Lemma 3.1.**

$$\prod_{i \in \mathbf{n}} \left( \sum_{j \in \mathbf{n}} b_{ij} \hat{x}_j \right) = \left( \prod_{i \in \mathbf{n}} \hat{x}_i \right) \text{per } \mathbf{B} \quad (4)$$

where  $\text{per } \mathbf{B}$  is the permanent of  $\mathbf{B}$ .

Let  $\hat{Y} = \{\hat{y}_1, \dots, \hat{y}_n\}$ . Define multiplication for the elements of  $\hat{Y}$  by

$$\hat{y}_i \hat{y}_j + \hat{y}_j \hat{y}_i = 0, \quad i, j = 1, \dots, n. \quad (5)$$

Then, it follows that

**Lemma 3.2.**

$$\prod_{i \in \mathbf{n}} \left( \sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j \right) = \left( \prod_{i \in \mathbf{n}} \hat{y}_i \right) \det \mathbf{B}. \quad (6)$$

Note that all basic properties of determinants are direct consequences of Lemma 3.2. Write

$$\sum_{j \in n} b_{ij} \hat{y}_j = \sum_{j \in n} b_{ij}^{(\lambda)} \hat{y}_j + (b_{ii} - \lambda_i) \hat{y}_i \hat{y} \quad (7)$$

where

$$b_{ii}^{(\lambda)} = \lambda_i, \quad b_{ij}^{(\lambda)} = b_{ij}, \quad i \neq j. \quad (8)$$

Let  $\mathbf{B}^{(\lambda)} = (b_{ij}^{(\lambda)})$ . By (6) and (7), it is straightforward to show the following result:

**Theorem 3.3.**

$$\det \mathbf{B} = \sum_{l=0}^n \sum_{I_l \subseteq n} \prod_{i \in I_l} (b_{ii} - \lambda_i) \det \mathbf{B}^{(\lambda)}(I_l | I_l), \quad (9)$$

where  $I_l = \{i_1, \dots, i_l\}$  and  $\mathbf{B}^{(\lambda)}(I_l | I_l)$  is the principal submatrix obtained from  $\mathbf{B}^{(\lambda)}$  by deleting its  $i_1, \dots, i_l$  rows and columns.

*Remark 3.1.* Let  $\mathbf{M}$  be an  $n \times n$  matrix. The convention  $\mathbf{M}(\mathbf{n} | \mathbf{n}) = 1$  has been used in (9) and hereafter.

Before proceeding with our discussion, we pause to note that Theorem 3.3 yields immediately a fundamental formula which can be used to compute the coefficients of a characteristic polynomial [9]:

**Corollary 3.4.** Write  $\det(\mathbf{B} - x\mathbf{I}) = \sum_{l=0}^n (-1)^l b_l x^l$ . Then

$$b_l = \sum_{I_l \subseteq n} \det \mathbf{B}(I_l | I_l). \quad (10)$$

Let

$$\mathbf{K}(t, t_1, \dots, t_n) = \begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \dots & \dots & \dots & \dots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix}, \quad (11)$$

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where

$$D_i = \sum_{j \in n} a_{ij} t_j, \quad i = 1, \dots, n. \quad (12)$$

Set

$$D(t_1, \dots, t_n) = \frac{\delta}{\delta t} \det \mathbf{K}(t, t_1, \dots, t_n) |_{t=1}.$$